On the principal eigenvalue of a Robin problem with a large parameter

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Dedicated to Viktor Borisovich Lidskiĭ on the occasion of his 80th birthday

17 January 2005

Abstract

We study the asymptotic behaviour of the principal eigenvalue of a Robin (or generalised Neumann) problem with a large parameter in the boundary condition for the Laplacian in a piecewise smooth domain. We show that the leading asymptotic term depends only on the singularities of the boundary of the domain, and give either explicit expressions or two-sided estimates for this term in a variety of situations.

1 Introduction

Let Ω be an open bounded set in $\mathbb{R}^m$ $(m \geq 1)$ with piecewise smooth, but not necessarily connected, boundary $\Gamma := \partial\Omega$. We investigate the spectral boundary value problem

$$-\Delta u = \lambda u \qquad \text{in } \Omega, \tag{1.1}$$

$$\frac{\partial u}{\partial n} - \gamma G u = 0 \qquad \text{on } \Gamma. \tag{1.2}$$

In (1.1), (1.2), $\frac{\partial}{\partial n}$ denotes the outward unit normal derivative, λ is the spectral parameter, γ is a positive parameter (which we later on assume to be large), and $G : \Gamma \to \mathbb{R}$ is a given continuous function. We will always assume that

$$\sup_{y\in\Gamma} G(y) > 0\,. \tag{1.3}$$

We treat the problem (1.1), (1.2) in the variational sense, associating it with the Rayleigh quotient

$$\mathcal{J}(v;\gamma,G) := \frac{\displaystyle\int_\Omega |\nabla v|^2 dx - \gamma \int_\Gamma G|v|^2 ds}{\displaystyle\int_\Omega |v|^2 dx}\,, \qquad v \in H^1(\Omega)\,,\ v \not\equiv 0\,. \tag{1.4}$$

For every fixed γ, the problem (1.1), (1.2) has a discrete spectrum of eigenvalues accumulating to $+\infty$. By

$$\Lambda(\Omega;\gamma,G) := \inf_{v\in H^1(\Omega)\,,\ v\not\equiv 0} \mathcal{J}(v;\gamma,G) \tag{1.5}$$

we denote the bottom of the spectrum of (1.1), (1.2).

Our aim is to study the asymptotic behaviour of $\Lambda(\Omega;\gamma,G)$ as $\gamma \to +\infty$ and its dependence upon the singularities of the boundary Γ.

The problem (1.1)–(1.2) naturally arises in the study of reaction-diffusion equation where a distributed absorption competes with a boundary source, see [2, 3] for details.

Remark 1.1. Sometimes, we shall also consider (1.1)–(1.2) for an *unbounded* domain Ω. In this case, we can no longer guarantee either the discreteness of the spectrum of (1.1)–(1.2), or its semi-boundedness below. We shall still use, however, the notation (1.5), allowing, in principle, for $\Lambda(\Omega;\gamma,G)$ to be equal to $-\infty$.

2 Basic properties of the principal eigenvalue

We shall mostly concentrate our attention on the case of constant boundary weight $G \equiv 1$; in this case, we shall denote for brevity

$$\mathcal{J}(v;\gamma) := \mathcal{J}(v;\gamma,1)\,, \qquad \Lambda(\Omega;\gamma) := \Lambda(\Omega;\gamma,1)\,.$$

See Remark 3.3 for the discussion of the case of an arbitrary smooth $G \not\equiv 1$.

We start with citing the following simple result of [3]:

Lemma 2.1. *For any bounded and sufficiently smooth $\Omega \subset \mathbb{R}^m$, $\Lambda(\Omega;\gamma)$ is a real analytic concave decreasing function of $\gamma \ge 0$, $\Lambda|_{\gamma=0} = 0$, and*

$$\left.\frac{d}{d\gamma}\Lambda(\Omega;\gamma)\right|_{\gamma=0} = -\frac{|\Gamma|_{m-1}}{|\Omega|_m}\,.$$

The problem (1.1)–(1.2) with $G \equiv 1$ admits a solution by separation of variables in several simple cases.

Example 2.2. For a ball $B_m(0,1) = \{|x| < 1\} \subset \mathbb{R}^m$, $\Lambda = \Lambda(B_m(0,1);\gamma)$ is given implicitly by

$$\sqrt{-\Lambda}\tanh\sqrt{-\Lambda} = \gamma\,, \qquad m = 1\,,$$
$$\sqrt{-\Lambda}\frac{I_{m/2}(\sqrt{-\Lambda})}{I_{m/2-1}(\sqrt{-\Lambda})} = \gamma\,, \qquad m \ge 2\,,$$

where I denotes a modified Bessel function. This implies that for any ball $B(a,R) := \{x : |x-a| < R\} \subset \mathbb{R}^m$,

$$\Lambda(B(a,R);\gamma) = -\gamma^2 + O(\gamma^2)\,, \qquad \gamma \to +\infty$$

(independently of the dimension m and radius R); it may be shown that the same asymptotics holds for an annulus $A_m(R_1,R) = \{|x| \in (R_1,R)\}$.

Example 2.3. For a parallelepiped $P(l_1,\dots,l_m) := \{|x_j| < l_j : j = 1,\dots,m\} \subset \mathbb{R}^m$ we get

$$\Lambda(P(l_1,\dots,l_m);\gamma) = -\sum_{j=1}^{m}\frac{\mu_j^2}{l_j^2},$$

where $\mu_j > 0$ solves a transcendental equation

$$\mu_j \tanh\mu_j = \gamma l_j\,.$$

Thus we obtain

$$\Lambda(P(l_1,\dots,l_m);\gamma) = -m\gamma^2 + O(\gamma^2)\,, \qquad \gamma \to +\infty\,.$$

Example 2.4. Let $\Omega = (0,+\infty)$, and $\Gamma = \{0\}$. It is easy to see that the bottom of the spectrum is an eigenvalue $\Lambda((0,+\infty);\gamma) = -\gamma^2$, the corresponding eigenfunction being $\exp(-\gamma x)$. Thus we arrive at a useful (and well-known) inequality

$$\int_0^\infty |v'(x)|^2\,dx - \gamma(v(0))^2 \ge -\gamma^2\int_0^\infty |v(x)|^2\,dx\,, \tag{2.1}$$

valid for all $v \in H^1((0,+\infty))$.

A slightly more complicated example is that of a planar angle $U_\alpha := \{z = x + iy \in \mathbb{C} : |\arg z| < \alpha\}$ of size 2α.

Example 2.5. Let $\Omega = U_\alpha$ with $\alpha < \pi/2$. Again the spectrum is not purely discrete; moreover, the separation of variable does not produce a complete set of generalised eigenfunctions. However, one can find an eigenfunction $u_0(x, y) = \exp(-\gamma x/\sin\alpha)$ and compute an eigenvalue $\lambda = -\gamma^2 \sin^{-2}\alpha$ explicitly. Thus $\Lambda(U_\alpha; \gamma) \le -\gamma^2 \sin^{-2}\alpha$. We shall now prove that this eigenvalue is in fact the bottom of the spectrum.

Lemma 2.6. *If $\alpha < \pi/2$,*

$$\Lambda(U_\alpha; \gamma) = -\gamma^2 \sin^{-2}\alpha\,. \tag{2.2}$$

Proof. It is sufficient to show that for all $v \in H^1(U_\alpha)$, we have

$$\int\limits_{U_\alpha} |\nabla v|^2 dz - \gamma \int\limits_{\partial U_\alpha} |v|^2 ds \ge -\gamma^2 (\sin^{-2}\alpha) \int\limits_{U_\alpha} |v|^2 dz\,. \tag{2.3}$$

As $ds = dy/\sin\alpha$, the left-hand side of (2.3) is bounded below by

$$\int dy \left(\int \left|\frac{\partial v}{\partial x}\right|^2 dx - \frac{\gamma}{\sin\alpha}|v|^2 \right)\,.$$

For each y the integrand is not smaller than $-\gamma^2(\sin^{-2}\alpha)\int |v|^2 dx$ by (2.1). Integrating over y gives (2.3). □

Example 2.7. Let us now consider the case of an angle U_α with $\alpha \in [\pi/2, \pi)$.

Lemma 2.8. *If $\alpha \ge \pi/2$,*

$$\Lambda(U_\alpha; \gamma) = -\gamma^2\,. \tag{2.4}$$

Proof. To prove an estimate above, we for simplicity consider a rotated angle $\widetilde{U}_\alpha := \{z = x + iy \in \mathbb{C} : 0 < \arg z < 2\alpha\}$. In order to get an upper bound $\Lambda(\widetilde{U}_\alpha; \gamma) \le -\gamma^2$, we construct a test function in the following manner. Let $\psi(s)$ be a smooth nonnegative function such that $\psi(s) = 1$ for $|s| < 1/2$, and $\psi(s) = 0$ for $|s| > 1$ Set now

$$\chi_\tau(s) = \begin{cases} 1\,, & \text{if } |s| < \tau - 1\,, \\ \psi(|s| - (\tau - 1))\,, & \text{if } \tau - 1 \le |s| < \tau\,, \\ 0\,, & \text{otherwise} \end{cases}$$

(a parameter τ is assumed to be greater than 1). Consider the function

$$v_\tau(x,y) = e^{-\gamma y}\chi_\tau(x\gamma - \tau)\,.$$

Then one can easily compute that

$$\begin{aligned}\mathcal{J}(v_\tau;\gamma) &= \gamma^2\left(-1 + \frac{\int_{-\infty}^{\infty}|\chi_\tau'(s)|^2\,ds}{\int_{-\infty}^{\infty}|\chi_\tau(s)|^2\,ds}\right)\\ &= \gamma^2\left(-1 + \frac{\int_{-1}^{1}|\psi'(s)|^2\,ds}{\int_{-1}^{1}|\psi(s)|^2\,ds + 2(\tau-1)}\right)\,,\end{aligned}$$

and therefore $\mathcal{J}(v_\tau;\gamma)\to -\gamma^2$ as $\tau\to\infty$. Thus, $\Lambda(U_\alpha;\gamma) = \Lambda(\widetilde{U}_\alpha;\gamma) \le -\gamma^2$.

To finish the proof, we need only to show that for $v\in H^1(U_\alpha)$,

$$\int\limits_{U_\alpha}|\nabla v|^2 dz - \gamma\int\limits_{\partial U_\alpha}|v|^2 ds \ge -\gamma^2\int\limits_{U_\alpha}|v|^2 dz\,. \tag{2.5}$$

Denote $V_\alpha = \{z : \alpha - \pi/2 < |\arg z| < \alpha\}\subset U_\alpha$. The estimate (2.5) will obviously be proved if we establish

$$\int\limits_{V_\alpha}|\nabla v|^2 dz - \gamma\int\limits_{\partial U_\alpha}|v|^2 ds \ge -\gamma^2\int\limits_{V_\alpha}|v|^2 dz\,.$$

But this can be done as in the proof of Lemma 2.6, by integrating first along ∂U_α, and then using one-dimensional inequalities (2.1) in the direction orthogonal to ∂U_α. ☐

We now consider a generalization of two previous examples to the multi-dimensional case.

Example 2.9. Let $K\subset\mathbb{R}^m = \{x : x/|x|\in M\}$ be a cone with the cross-section $M\subset S^{m-1}$. Any homothety $f : x\mapsto ax$ ($x\in\mathbb{R}^m$, $a>0$) maps K onto itself. Then, as easily seen by a change of variables $w = \gamma^{-1}x$,

$$\Lambda(K;\gamma) = \gamma^2\Lambda(K;1)\,. \tag{2.6}$$

In particular, if K contains a half-space, then, repeating the argument of Lemma 2.8 with minor adjustments, one can show that $\Lambda(K,1) = -1$ and so

$$\Lambda(K;\gamma) = -\gamma^2\,. \tag{2.7}$$

All the above examples suggest that in general one can expect

$$\Lambda(\Omega;\gamma) = -C_\Omega\gamma^2 + O(\gamma^2)\,, \qquad \gamma\to+\infty\,. \tag{2.8}$$

Some partial progress towards establishing (2.8) was already achieved in [3]. In particular, the following Theorems were proved.

Theorem 2.10. *Let $\Omega\subset\mathbb{R}^m$ be a domain with piecewise smooth boundary Γ. Then*

$$\limsup_{\gamma\to+\infty}\frac{\Lambda(\Omega;\gamma,1)}{\gamma^2}\le -1.$$

Theorem 2.11. *Let $\Omega\subset\mathbb{R}^m$ be a domain with smooth boundary $\partial\Omega$. Then*

$$\Lambda(\Omega;\gamma) = -\gamma^2(1+o(1))\,, \qquad \gamma\to+\infty\,.$$

Remark 2.12. The actual statements in [3] are slightly weaker than the versions above, but the proofs can be easily modified. Note that the proof of Theorem 2.10 can be done by constructing a test function very similar to the one used in the proof of Lemma 2.8.

The situation, however, becomes more intriguing even in dimension two, if Γ is not smooth. Suppose that $\Omega\subset\mathbb{R}^2$ is a planar domain with n corner points $y_1,\dots,y_n$ on its boundary Γ. The following conjecture was made in [3]:

Conjecture 2.13. Let $\Omega\subset\mathbb{R}^2$ be a planar domain with n corner points $y_1,\dots,y_n$ on its boundary Γ and let α_j, $j=1,\dots,n$ denote the inner half-angles of the boundary at the points y_j. Assume that $0<\alpha_j<\frac{\pi}{2}$. Then (2.8) holds with

$$C_\Omega = \max_{j=1,\dots,n}\left\{\sin^{-2}(\alpha_j)\right\}\,.$$

This conjecture was proved in [3] only in the model case when Ω is a triangle.

As we shall see later on, formula (2.8) does not, in general, hold if we allow Γ to have zero angles (i.e., outward pointing cusps, see Example 3.4). We shall thus restrict ourselves to the case when Ω is piecewise smooth in a suitable sense, see below for the precise definition. Under this assumption, we first of all prove that the asymptotic formula (2.8) holds. Moreover, we compute C_Ω explicitly in the planar case, thus proving Conjecture 2.13. In the case of dimension $m\ge 3$, we give some upper and lower bounds on C_Ω, which, in some special cases, amount to a complete answer.

3 Reduction to the boundary

We shall only consider the case when Ω is piecewise smooth in the following sense: for each point $y \in \Gamma$ there exists an infinite "model" cone K_y such that for a small enough ball $B(y,r)$ of radius r centred at y there exists an infinitely smooth diffeomorphism $f_y : K_y \cap B(0,r) \to \Omega \cap B(y,r)$ with $f_y(0) = y$ and the derivative of f_y at 0 being the identity matrix (we shall write in this case that $\Omega \sim K_y$ near a point $y \in \Gamma$). For example, if y is a regular point of Γ, then K_y is a half-space.

We require additionally that Ω satisfies the uniform interior cone condition [1], i.e. there exists a fixed cone K with non-empty interior such that each K_y contains a cone congruent to K. (See Example 3.4 for a discussion of a case where this condition fails.)

Definition 3.1. Let $\Omega \sim K_y$ near a point $y \in \Gamma$. We denote $C_y := -\Lambda(K_y; 1)$.

Our main result indicates that the asymptotic behaviour of $\Lambda(\Omega; \gamma, 1)$ is in a sense "localised" on the boundary.

Theorem 3.2. *Let Ω be piecewise smooth in the above sense and satisfy the uniform interior cone condition. Then*

$$\Lambda(\Omega;\gamma) = -\gamma^2 \sup_{y\in\Gamma} C_y + o(\gamma^2)\,, \qquad \gamma \to +\infty\,. \tag{3.1}$$

Remark 3.3. This result can be easily generalised for the case of our original setting of a non-constant boundary weight $G(y)$ satisfying (1.3):

$$\Lambda(\Omega;\gamma, G) = -\gamma^2 \sup_{\substack{y\in\Gamma\\ G(y)>0}} \{G(y)^2 C_y\} + o(\gamma^2)\,, \qquad \gamma \to +\infty\,.$$

Example 3.4. Formula (2.8) does not, in general, hold if Γ is allowed to have outward pointing cusps. In particular, for a planar domain

$$\Upsilon_p = \{(x,y) \in \mathbb{R}^2 : x > 0\,,\ |y| < x^p\}\,, \qquad p > 1$$

one can show that

$$\Lambda(\Upsilon_p;\gamma) \le -\operatorname{const} \begin{cases} \gamma^{2/(2-p)} & \text{for } 1 < p < 2\,,\\ \gamma^N \quad \text{with any } N > 0 & \text{for } p \ge 2\,,\end{cases}$$

by choosing the test function $v = \exp\left(-\gamma x^{q_p}\right)$ with $q_p = 2 - p$ for $1 < p < 2$ and $q_p = 2$ for $p \ge 2$.

In order to provide the explicit asymptotic formula for $\Lambda(\Omega;\gamma)$ in the piecewise smooth case it remains to obtain the information on the dependence of the constants C_y upon the local geometry of Γ at y.

It is easy to do this, firstly, in the case of a regular boundary in any dimension, and, secondly, in the two-dimensional case, where the necessary information is already contained in Lemmas 2.6 and 2.8.

Theorem 3.5. *Let Γ be smooth at y. Then $C_y = 1$.*

Moreover, $C_y = 1$ whenever there exists an $(m-1)$-dimensional hyperplane H_y passing through y such that for small r, $B(y,r) \cap H_y \subset \overline{\Omega}$.

Theorem 3.6. *Let $\Omega \subset \mathbb{R}^2$ and let $y \in \Gamma$ be such that $\Omega \sim U_\alpha$ near y. Then*

$$C_y = \begin{cases} 1\,, & \text{if } \alpha \geq \pi/2\,;\\ \sin^{-2}\alpha\,, & \text{if } \alpha \leq \pi/2\,. \end{cases}$$

Theorems 3.2, 3.5, and 3.6 prove the validity of Conjecture 2.13.

In more general cases, we are only able to provide the two-sided estimates on C_y, and obtain the precise formulae only under rather restrictive additional assumptions. These results are collected in Section 5.

4 Proof of Theorem 3.2

We proceed via a sequence of auxiliary Definitions and Lemmas.

Definition 4.1. Let $K \subset \mathbb{R}^m$ be a cone with cross-section $M \subset S^{m-1}$, and let $r > 0$. By $\mathfrak{K}_r = \mathfrak{K}_r(K)$ we denote the family of "truncated" cones $K_{r,R}$ such that

$$K_{r,R} = \{x \in \mathbb{R}^m : \theta := x/|x| \in M \subset S^{m-1}\,,\ |x| < rR(\theta)\}\,,$$

where $R : M \to [1, m]$ is a piecewise smooth function. Thus, for any $K_{r,R} \in \mathfrak{K}_r$ we have

$$K \cap B(0,r) \subset K_{r,R} \subset K \cap B(0,mr)\,.$$

Let $K_{r,R} \in \mathfrak{K}_r$, and let $\sharp$ be an index assuming values D or N (which in turn stand for Dirichlet or Neumann boundary conditions). By $\Lambda^\sharp(K_{r,R};\gamma)$ we denote the bottom of the spectrum of the boundary value problem (1.1) considered in $K_{r,R}$ with boundary conditions (1.2) on $\partial K_{r,R} \cap \partial K = \{x \in \partial K_{r,R} : x/|x| \in \partial M\}$ and with the boundary condition defined by $\sharp$ on the rest of the boundary $\{x : x/|x| \in M\,,\ |x| = R(\theta)\}$ (this boundary value problem is of course considered in the variational sense).

It is important to note that a simple change of variables as in Example 2.9 leads to the re-scaling relations

$$\Lambda^\sharp(K_{r,R};\gamma) = \gamma^2 \Lambda^\sharp(K_{r\gamma,R};1)\,. \tag{4.1}$$

These formulae show that the bottoms of the spectra $\Lambda^\sharp(K_{r,R};\gamma)$ are determined (modulo a multiplication by γ^2) by a single parameter $\mu := r\gamma$ via $\Lambda^\sharp(K_{\mu,R};1)$. It is therefore the latter which we proceed to study.

The first Lemma gives a relation between the bottoms of the spectra for an infinite cone K and its finite "cut-offs".

Lemma 4.2. *Let $K_{r,R} \in \mathfrak{K}_r(K)$ and let $\mu = r\gamma$. Then, as $\mu \to \infty$,*

$$\frac{\Lambda^\sharp(K_{r,R};\gamma)}{\gamma^2} = \Lambda(K;1) + o(1)\,.$$

Proof of Lemma 4.2. By (4.1), we need to prove that

$$\lim_{\mu\to\infty} \Lambda^\sharp(K_{\mu,R};1) = \Lambda(K;1)\,.$$

This can be done by considering a function $v \in H^1(K)$ and comparing the Rayleigh quotients $J(v;1)$ with "truncated" quotients $J(v\psi(\cdot/\mu);1)$, where ψ is the same as in the proof of Lemma 2.8. An easy but somewhat tedious computation which we omit shows that as $\mu \to +\infty$, we have $J(v\psi(\cdot/\mu);1) \to J(v;1)$, which finishes the proof. ☐

Let $y \in \Gamma$, and let K_y be a cone with cross-section M such that $\Omega \sim K_y$ near y. Let $r > 0$ and $K_{y,r,R} \in \mathfrak{K}_r(K_y)$. We define $\Omega_{y,r,R} := f_y(K_{y,r,R})$, and introduce the numbers $\Lambda^\sharp(\Omega_{y,r,R};\gamma)$ similarly to $\Lambda^\sharp(K_{y,r,R};\gamma)$.

Lemma 4.3. *Let $\mu > 0$ be fixed. Then*

$$\lim_{r\to+0} \frac{\Lambda^\sharp(K_{y,r,R};\mu/r)}{\Lambda^\sharp(\Omega_{y,r,R};\mu/r)} = 1 \tag{4.2}$$

uniformly over $y \in \Gamma$.

Proof of Lemma 4.3. Let us denote by $\widetilde{\Omega}_{y,r,R}$ an image of $\Omega_{y,r,R}$ under the homothety $h_{y,r} : z \mapsto y + r^{-1}(z-y)$. Conditions imposed on the mapping f_y imply that as $r \to +0$, $\widetilde{\Omega}_{y,r,R} \to K_{y,1,R}$ in the following sense. The volume element of $\widetilde{\Omega}_{y,r,R}$ at a point $(h_{y,r} \circ f_y)(xr)$ tends to the volume element of $K_{y,1,R}$ at point x, and the analogous statement holds for the area element of the boundary. Since μ is fixed, this implies that the bottoms of the spectra $\Lambda^\sharp(\widetilde{\Omega}_{y,r,R};\mu)$ (with boundary conditions as described above) converge to $\Lambda^\sharp(K_{y,1,R};\mu)$ as $r \to +0$. Now the same re-scaling arguments as before imply (4.2). A simple compactness argument shows that this convergence is uniform in $y \in \Gamma$. ☐

Remark 4.4. It is easy to see that the estimates of Lemmas 4.2 and 4.3 are uniform in R if we assume that all first partial derivatives of R are bounded by a given constant.

We can now conclude the proof of Theorem 3.2 itself. First of all, given an arbitrary positive ϵ and $y \in \Gamma$, we use Lemma 4.2 to find a positive $\mu(y)$ such that

$$|\gamma^{-2}\Lambda^\sharp(K_{y,r,R};\gamma) + C_y| < \frac{\epsilon}{2}\,, \tag{4.3}$$

whenever $\gamma \geq r^{-1}\mu(y)$. It is easy to see that $\mu(y)$ can be chosen to be continuous on each smooth component of the boundary. Therefore, there exists $\widetilde{\mu} = \sup\limits_{y\in\Gamma} \mu(y)$. Let us fix this value of $\widetilde{\mu}$ for the rest of the proof.

Formula (3.1) splits into two asymptotic inequalities. The inequality

$$\Lambda(\Omega;\gamma) \leq -\gamma^2 \sup_{y\in\Gamma} C_y + \epsilon\gamma^2\,, \qquad \gamma \to +\infty$$

follows immediately from formula (4.3), Lemma 4.3 (with $\sharp = D$ and $\mu = \widetilde{\mu}$) and the obvious inequality

$$\Lambda(\Omega;\gamma) \leq \Lambda^D(\Omega_{y,r,R};\gamma)\,.$$

In order to prove the opposite inequality

$$\Lambda(\Omega;\gamma) \geq -\gamma^2 \sup_{y\in\Gamma} C_y + \epsilon\gamma^2\,, \qquad \gamma \to +\infty\,, \tag{4.4}$$

we consider a partition $\overline{\Omega} = \overline{\bigsqcup_{\ell=0}^N Q_\ell}$ by disjoint sets Q_ℓ satisfying the following properties: $Q_0 \Subset \Omega$ (i.e. $Q_0 \cap \Gamma = \emptyset$), and for each $\ell \geq 1$, $Q_\ell = \Omega_{y,r,R} = f_y(K_{r,R})$ with some $r > 0$, $y \in \Gamma$, and $K_{r,R} \in \mathfrak{K}_r(K_y)$, such that $\Omega \sim K_y$ near y. Such a partition can be constructed for each sufficiently small $r > 0$ by considering, for example, a partition of $\mathbb{R}^m$ into cubes of size r, and including into Q_0 all the cubes which lie strictly inside Ω. Note that $\Gamma = \overline{\bigcup_{\ell=1}^N (\Gamma \cap Q_\ell)}$.

Now we use the following inequality: assuming that $J(v;\gamma)$ is negative for

some $v \in H^1(\Omega) \setminus \{0\}$, we have

$$
\begin{aligned}
J(v;\gamma) &= \frac{\displaystyle\int\limits_{\Omega} |\nabla v|^2 dx - \gamma \int\limits_{\Gamma} |v|^2 ds}{\displaystyle\int\limits_{\Omega} |v|^2 dx} \ge \frac{\displaystyle\int\limits_{\Omega\setminus Q_0} |\nabla v|^2 dx - \gamma \int\limits_{\Gamma} |v|^2 ds}{\displaystyle\int\limits_{\Omega\setminus Q_0} |v|^2 dx} \\
&= \frac{\displaystyle\sum_{\ell=1}^{N} \int\limits_{Q_\ell} |\nabla v|^2 dx - \gamma \sum_{\ell=1}^{N} \int\limits_{\Gamma\cap Q_\ell} |v|^2 ds}{\sum_{\ell=1}^{N} \displaystyle\int\limits_{Q_\ell} |v|^2 dx} \\
&\ge \min_{\ell=1\dots N} \frac{\displaystyle\int\limits_{Q_\ell} |\nabla v|^2 dx - \gamma \int\limits_{\Gamma\cap Q_\ell} |v|^2 ds}{\displaystyle\int\limits_{Q_\ell} |v|^2 dx}\,.
\end{aligned}
\tag{4.5}
$$

Note that the last expression in (4.5) is bounded below by $\inf \Lambda^N(\Omega_{y,r,R};\gamma)$, where the infimum is taken over all $y \in \Gamma$ and all functions R admissible in the sense of Definition 4.1.

Finally, taking the size of the partition $r \to +0$, and using formula (4.3) and Lemma 4.2 with $\mu = \widetilde{\mu}$, we obtain (4.4).

5 Estimates in the general case

Let us now discuss the general case. As we have already shown, the problem of computing the constant $C_\Omega = \sup\limits_{y\in\Gamma} C_y$ in (2.8) is reduced to calculating the bottoms of the spectra $\Lambda(K_y;1) = -C_y$ for infinite model cones K_y. We have also shown that $C_y = 1$ when Γ is smooth at y. We now consider a case when Γ is singular at y.

Let j be the co-dimension of a singularity of Γ at y. By this we mean that $K_y = \mathbb{R}^{m-j} \times \widetilde{K}$, with $\widetilde{K} = \{z \in \mathbb{R}^j : z/|z| \in \widetilde{M}\}$, with the singular cross-section $\widetilde{M} \subset S^{j-1}$. If $j \ge 3$, we restrict our analysis to the case when the closure of $\widetilde{M}$ is contained in open hemisphere $\{\theta \in S^{j-1} : \theta_1 > 0\}$. For simplicity, we assume that $\widetilde{M}$ is convex (this stronger requirement may be relaxed, see Remark 5.2).

The case $j=1$ corresponds to a regular point $y\in\Gamma$. The case $j=2$ is treated in exactly the same way as the planar case, as in this situation $\widetilde{K}=U_\alpha$ and the constant C_y is the same as in Theorem 3.6.

Consider now the case $j\ge 3$. It might seem natural to introduce the spherical coordinates on $\widetilde{K}$ at this stage. Unfortunately, such an approach is not likely to succeed — although the variables separate, the resulting lower-dimensional problems are coupled in a complicated way. Indeed, Example 2.5 shows that the principal eigenfunction is not easily expressed in spherical coordinates. Therefore, we will try to choose a coordinate frame more suitable for this problem. Once more, Example 2.5 gives us a helpful insight into what this coordinate frame should be.

We need more notation. Let $w\in\widetilde{K}$ with $\theta=w/|w|\in\widetilde{M}$. We define Π_θ as a $(j-1)$-dimensional hyperplane passing through θ and orthogonal to w. Let $P_\theta=\Pi_\theta\cap\partial\widetilde{K}$. We need to consider only the points θ such that P_θ is bounded and $\theta\in P_\theta$. Such directions θ always exist due to the convexity of $\widetilde{M}$.

We now introduce the coordinates $(\xi,\eta)\in\mathbb{R}\times\mathbb{R}^{j-1}$ of a point $z\in\mathbb{R}^j$, such that $\xi=z\cdot\theta$ is a coordinate along θ and $\eta=z-\xi\theta$ represent coordinates along the plane Π_θ.

We also need the spherical coordinates (ρ,φ) with the origin at θ on Π_θ, such that $\rho=|\eta|$ and $\varphi=\eta/|\eta|\in S^{j-2}$. We define a function $b(\varphi)=b_\theta(\varphi)$ in such a way that $P_\theta=\{(\rho,\varphi):\rho=b(\varphi)\}$.

In these coordinates,

$$\widetilde{K}=\{(\xi,\rho,\varphi):\xi>0\,,\ \rho<\xi b_\theta(\varphi)\}\tag{5.1}$$

and

$$\partial\widetilde{K}=\{(\xi,\rho,\varphi):\xi>0\,,\ \rho=\xi b_\theta(\varphi)\}\,.\tag{5.2}$$

Denote

$$\sigma_\theta(\varphi):=\sqrt{1+b_\theta^{-2}(\varphi)+(b'_\theta(\varphi))^2 b_\theta^{-4}(\varphi)}\,.\tag{5.3}$$

We are ready now to formulate a general statement in the case $j=3$.

Theorem 5.1. *Let $y\in\Gamma$ be a singular point of co-dimension three in the above sense. Then the constant C_y satisfies the following two-sided estimates:*

$$\sup_\theta\left(\frac{\displaystyle\int_{S^1} b_\theta^2(\varphi)\sigma_\theta(\varphi)\,d\varphi}{\displaystyle\int_{S^1} b_\theta^2(\varphi)\,d\varphi}\right)^2\le C_y\le\inf_\theta\sup_\varphi\sigma_\theta^2(\varphi)\tag{5.4}$$

Remark 5.2. Theorem 5.1 can be extended to the case of non-convex $\widetilde{M}$. Then, the function $b_\theta(\varphi)$ (which defines the boundary) may become multivalued. In that case we need to treat the integrals in the left-hand side of (5.4) separately along each branch of b_θ, and count them with a plus or minus sign.

Proof of Theorem 5.1. The separation of variables shows that $C_y = -\Lambda(\widetilde{K};1)$. We start by estimating C_y below (and thus $\Lambda(\widetilde{K};1)$ above). Let us fix $\theta \in \widetilde{M}$ satisfying the above conditions; for brevity we shall omit the subscript θ in all the intermediate calculations.

Consider the following test function

$$v(z) = \exp(-a\xi)\,, \qquad z = (\xi,\rho,\varphi) \in \widetilde{K}\,, \tag{5.5}$$

where a is a positive parameter to be chosen later.

Then we explicitly calculate

$$\int\limits_{\widetilde{K}} v^2(z)\,dz = \int\limits_0^\infty \exp(-2a\xi)\,d\xi \int\limits_{S^1} d\varphi \int\limits_0^{\xi b(\varphi)} \rho d\rho = \frac{1}{8a^3}\int\limits_{S^1} b^2(\varphi)\,d\varphi \tag{5.6}$$

and

$$\int\limits_{\widetilde{K}} |\nabla v(z)|^2\,dz = a^2 \int\limits_{\widetilde{K}} v^2(z)\,dz = \frac{1}{8a}\int\limits_{S^1} b^2(\varphi)\,d\varphi\,. \tag{5.7}$$

Let us now calculate the integral along the boundary $\partial\widetilde{K}$. For each $\widetilde{\eta} = (\widetilde{\rho},\widetilde{\varphi}) \in \mathbb{R}^2$ there exists a unique point $z = (\widetilde{\xi},\widetilde{\eta}) = (\widetilde{\xi}(\widetilde{\eta}),\widetilde{\eta}) \in \partial\widetilde{K}$, where one can easily compute $\widetilde{\xi}(\widetilde{\eta}) = \dfrac{\widetilde{\rho}}{b(\widetilde{\varphi})}$. Thus the area element of the boundary ds can be expressed as $\dfrac{1}{\cos\beta}d\widetilde{\eta}$, where β is an angle between two planes. One of these planes is Π_θ and the other one is the plane containing the origin and the straight line L which lies in Π_θ and is tangent to P_θ at the point $\xi = 1$, $\rho = b(\widetilde{\varphi})$, $\varphi = \widetilde{\varphi}$. Without loss of generality we assume now that $\widetilde{\varphi} = 0$, otherwise we just rotate the picture. Then the equation of L in cartesian coordinates $\eta = (\eta_1,\eta_2)$ on P_θ becomes $L = \{\eta_1 = b(0) + tb'(0)\,,\ \eta_2 = b(0)t : t \in \mathbb{R}\}$. It is a simple geometric exercise to show that the base of the perpendicular dropped from the origin onto L corresponds to the parameter value $t^* = -\dfrac{b(0)b'(0)}{b(0)^2 + (b'(0))^2}$ and therefore this base point is given by $(\eta_1^*,\eta_2^*) = \dfrac{b(0)^2}{b(0)^2 + (b'(0))^2}(b(0),-b'(0))$. Another geometric exersise shows that $\cot\beta$ is equal to the length of the vector

(η_1^*, η_2^*), and therefore

$$\frac{1}{\cos\beta} = \sqrt{1+\cot^{-2}\beta} = \sqrt{1+b^{-2}(0)+(b'(0))^2 b^{-4}(0)}\,.$$

Thus, the area element, with account of (5.3), is

$$ds = \frac{1}{\cos\beta} d\widetilde{\eta} = \sqrt{1+b^{-2}(\widetilde{\varphi})+(b'(\widetilde{\varphi}))^2 b^{-4}(\widetilde{\varphi})}\, d\widetilde{\eta} = \sigma(\widetilde{\varphi})\, d\widetilde{\eta}\,, \tag{5.8}$$

and we can evaluate the boundary contribution as

$$\begin{aligned}
\int\limits_{\partial\widetilde{K}} v^2(z)\, ds &= \int\limits_{\mathbb{R}^2} \exp(-2a\widetilde{\xi})\sigma(\widetilde{\varphi})\, d\widetilde{\eta} \\
&= \int\limits_{S^1} d\widetilde{\varphi}\,\sigma(\widetilde{\varphi}) \int\limits_0^\infty \widetilde{\rho}\exp(-2a\widetilde{\rho}/b(\widetilde{\varphi}))\, d\widetilde{\rho} \\
&= \frac{1}{4a^2}\int\limits_{S^1} b^2(\varphi)\sigma(\varphi)\, d\varphi\,.
\end{aligned} \tag{5.9}$$

Combining now (5.6), (5.7), and (5.9), we obtain

$$J(v;1) = a^2 - \frac{2a\displaystyle\int\limits_{S^1} b^2(\varphi)\sigma(\varphi)\, d\varphi}{\displaystyle\int\limits_{S^1} b^2(\varphi)\, d\varphi}\,.$$

Optimising with respect to a gives

$$a = \frac{\displaystyle\int\limits_{S^1} b_\theta^2(\varphi)\sigma(\varphi)\, d\varphi}{\displaystyle\int\limits_{S^1} b_\theta^2(\varphi)\, d\varphi}\,, \tag{5.10}$$

and further optimization with respect to θ produces the desired lower bound in (5.4).

Let us now prove the upper bound on C_y in (5.4), which corresponds to the lower bound on $\Lambda(\widetilde{K};1)$. We need to show that for any $v \in H^1(\widetilde{K})$ and any $\theta \in \widetilde{M}$ the following inequality holds:

$$\int\limits_{\widetilde{K}} |\nabla v(z)|^2\, dz - \int\limits_{\partial\widetilde{K}} |v(z)|^2\, ds \ge -\left(\sup_\varphi \sigma(\varphi)\right)^2 \int\limits_{\widetilde{K}} |v(z)|^2\, dz\,. \tag{5.11}$$

Using the obvious estimate

$$\int_{\widetilde{K}} |\nabla v|^2\,dz \geq \int_{\widetilde{K}} |\partial_\xi v|^2\,dz\,,$$

formula (5.8) for the area element, and inequality (2.1) in the variable ξ for each value of η, we arrive at (5.11). This completes the proof. $\square$

Remark 5.3. In the case of a three-edged corner (i.e. when $\widetilde{M}$ is a two-dimensional spherical triangle) the left- and right-hand sides of (5.4) in fact coincide, so Theorem 5.1 gives the exact expression for C_y. The same is true if $\widetilde{M}$ is a spherical polygon which has an inscribed circle (i.e., a circle touching all the sides of $\widetilde{M}$). Indeed, in this case the supremum in the left-hand side and the infimum in the right-hand side of (5.4) are equal and are attained when θ is the centre of the inscribed circle. This immediately follows from the fact that in this case and for this choice of θ, $\sigma \equiv \mathrm{const}$. Moreover, it is easy to see that the test function (5.5) with the parameter a given by (5.10) is an eigenfunction with the eigenvalue at the bottom of the spectrum $\Lambda(\widetilde{K};1)$.

Thus, Theorems 3.5, 3.6, and 5.1 provide an exact asymptotics of $\Lambda(\Omega;\gamma)$ whenever $m=3$ and each vertex of Ω has three edges coming from it.

Assume now that $j>3$. This case is pretty much similar to the previous one (in particular, the test function used in obtaining the estimate below on C_y is still given by (5.5)), the only difference being that the area element of the boundary now becomes a volume element of co-dimension one and is much more cumbersome to calculate. We skip the detailed calculations.

In order to state the result, we need more notation. Define a $(j-2)$-dimensional vector $\zeta_\theta(\varphi) := b_\theta(\varphi)\nabla_\varphi b_\theta(\varphi)$ and $(j-2)\times(j-2)$ matrix $Z_\theta(\varphi) := b_\theta^2(\varphi)I + (\nabla_\varphi\otimes\nabla_\varphi)b_\theta(\varphi)$. Now put $\Psi_\theta(\varphi) := (Z_\theta^{-1}(\varphi)\zeta_\theta(\varphi))$ and

$$\Sigma_\theta(\varphi) := \sqrt{1 + \left((b_\theta(\varphi) - \Psi_\theta(\varphi)\cdot\nabla_\varphi b_\theta(\varphi))^2 + b_\theta^2(\varphi)|\Psi_\theta(\varphi)|^2\right)^{-1}}\,. \tag{5.12}$$

Theorem 5.4. *Let $y\in\Gamma$ be a singular point of co-dimension $j\geq 4$ in the above sense. Then the constant C_y satisfies the following two-sided estimates:*

$$\sup_\theta \left(\frac{\displaystyle\int_{S^{j-2}} b_\theta^{j-1}(\varphi)\Sigma_\theta(\varphi)\,d\varphi}{\displaystyle\int_{S^{j-2}} b_\theta^{j-1}(\varphi)\,d\varphi} \right)^2 \leq C_y \leq \inf_\theta \sup_\varphi \Sigma_\theta^2(\varphi)\,. \tag{5.13}$$

Remark 5.5. It is easily seen that Theorem 5.1 is in fact a partial case of Theorem 5.4 if we formally set $j = 3$ in the latter. Indeed, for $j = 3$ all the quantities depend upon a scalar parameter φ, and we obtain

$$\zeta_\theta(\varphi) = b_\theta(\varphi) b'_\theta(\varphi)\,, \quad Z_\theta(\varphi) = b_\theta^2(\varphi) + (b'_\theta(\varphi))^2\,, \quad \Psi_\theta(\varphi) = \frac{b_\theta(\varphi) b'_\theta(\varphi)}{b_\theta^2(\varphi) + (b'_\theta(\varphi))^2}\,,$$

giving

$$\begin{aligned}\Sigma_\theta(\varphi) &= \sqrt{1 + \left(\left(b_\theta(\varphi) - \frac{b_\theta(\varphi)(b'_\theta(\varphi))^2}{b_\theta^2(\varphi) + (b'_\theta(\varphi))^2}\right)^2 + b_\theta^2(\varphi)\frac{(b'_\theta(\varphi))^2}{(b_\theta^2(\varphi) + (b'_\theta(\varphi))^2)^2}\right)^{-1}} \\ &= \sqrt{1 + \frac{b_\theta^2(\varphi) + (b'_\theta(\varphi))^2}{b_\theta^4(\varphi)}} = \sigma_\theta(\varphi)\,,\end{aligned}$$

so that formula (5.13) becomes (5.4).

Remark 5.6. As before, the estimates (5.13) give the precise value of C_y whenever M is a $(j-1)$-dimensional spherical polyhedron which admits an inscribed ball (for example when M has exactly j faces). Moreover, the bottom of the spectrum is again an eigenvalue corresponding to the eigenfunction (5.5).

Acknowledgements: our collaboration was partially supported by the EPSRC Spectral Theory Network.

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